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OSCILLATION CRITERIA FOR A NONLINEAR CONFORMABLE FRACTIONAL DIFFERENTIAL SYSTEM WITH A FORCING TERM

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Abstract

We employ the averaging functions, conformable fractional derivative and some inequalities to establish new oscillatory behaviour of the solutions of fractional differential system with a forcing term. The results obtained here extend and improve on some existing results. Examples are also given to show the validity of our results.

1. Introduction

The study of oscillation of differential equation is one of the traditional trends in the qualitative theory of differential equations [see 2, 8, 13, 15]; however, in the last two decades, it has been transfixed with fractional differential equations by many authors. This is due to its numerous applications in science and engineering.

Thus, the oscillation of fractional differential equation has increased substantially using different approaches such as Caputo, Riemann-Liouville,

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modified Riemann-Liouville [3, 12, 16] and conformable fractional derivative [4, 5, 11].

Many authors have worked on the oscillation of two-dimensional linear and nonlinear differential systems [7, 9, 10, 18] using different methods. To the best of our knowledge, little or no work has been done on nonlinear conformable fractional differential system.

Our interest here is to study the oscillation of the following nonlinear conformable fractional differential system with a forcing term:

$$\left.\begin{aligned} &D^{\alpha}(u(t)) = a(t)f(v(t)) \\ &D^{\alpha}(v(t)) = -b(t)g\left(\int_0^t u'(s)f'(u(s))ds\right) + r(t) \\ &0 < \alpha < 1, \quad t \geq t_0 > 0 \end{aligned}\right\}, \tag{1}$$

where D^{α} denotes conformable fractional derivative of order α w.r.t. t.

Throughout this paper, we assume that the following conditions hold:

(I) $a(t) \in C^{\alpha}([t_0, \infty), (0, \infty))$ with the condition

$$\int_{t_0}^{\infty} \frac{a(s)}{s^{1-\alpha}} ds = \infty;$$

(II) $b(t), r(t) \in C([t_0, \infty), [0, \infty))$ such that $b(t), r(t)$ are not identically zeros on any interval of the form $[T_0, \infty)$, where $T_0 \geq t_0$;

(III) $g, f \in C(\Re, \Re)$ with $ug(u), vf(v) > 0$; $0 < g'(K(t)) \leq M$, $0 < f'(v(t)) \leq m$, $\frac{1}{g(K)} \leq c$; $cb(t) > r(t)$; for $u, v, g(K) \neq 0$ and

$$K(t) = \int_0^t u'(s)f'(u(s))ds.$$

A solution $(u(t), v(t))$ to the system (1) is oscillatory if it has arbitrarily large zeros, and it is nonoscillatory otherwise. System (1) is said to be oscillatory if all its solutions are oscillatory, otherwise it is nonoscillatory.

2. Preliminaries

In this section, we ought to explain the basic concept of conformable fractional derivative, but we refer the readers who are not familiar with conformable fractional derivatives to see ([1, 6]) for more details.

Lemma 1. *Suppose* $\psi'(y(s)) \leq d$ *and let*

$$K(t) = \int_0^t y'(s)\psi'(y(s))ds.$$

Then,

$$K'(t) \leq \frac{dD^{\alpha}y(t)}{t^{1-\alpha}}.$$

Lemma 2. *Suppose* $a(t) \geq 0$. *Then the first component* $u(t)$ *of a nonoscillatory solution* $(u(t), v(t))$ *of* (1) *is also nonoscillatory*.

Proof. The proof follows from Lemma 7.2.1 [14]. □

3. Main Result

In this section, we use the Riccati technique to establish sufficient conditions for system (1) to be oscillatory.

Theorem 1. *Suppose that the assumptions I-III hold. Assume also that* ∃ *a positive function* $\delta \in C^{\alpha}([0, \infty), \Re_{+})$ *such that*

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{m\delta(s)}{s^{1-\alpha}}(b(s) - cr(s)) - s^{1-\alpha}\left(\frac{\delta'}{2\sqrt{(dMa(s)\delta(s))}} \right)^2 \right] ds = \infty. \quad (2)$$

Then every solution of (1) *is oscillatory*.

Proof. Suppose that (1) has a nonoscillatory solution of $(u(t), v(t))$ on $[t_0, \infty)$. From Lemma 2, $u(t)$ is always nonoscillatory. Without loss of generality, we assume that $u(t) > 0$ for $t \geq t_1 \geq t_0$ and $K(t) > 0$ for $t \geq t_1$.

From (1), we have the following inequality,

$$f(v(t)) = \frac{1}{a(t)} D^{\alpha}(u(t))$$

$$D^{\alpha}\left[\frac{1}{a(t)} D^{\alpha}(u(t))\right] - f'(v(t)) D^{\alpha} v(t) = 0$$

$$D^{\alpha}\left[\frac{1}{a(t)} D^{\alpha}(u(t))\right] - m(-b(t)g(K) + r(t)) \le 0$$

$$D^{\alpha}\left[\frac{1}{a(t)} D^{\alpha}(u(t))\right] + m(b(t)g(K) - r(t)) \le 0$$

which implies that

$$D^{\alpha}\left[\frac{1}{a(t)} D^{\alpha}(u(t))\right] < 0, \quad t \ge t_1.$$

Thus,

$$D^{\alpha}u(t) \ge 0 \text{ or } D^{\alpha}u(t) < 0, \quad t \ge t_1.$$

We claim that $D^{\alpha}u(t) \ge 0$ for $t \ge t_1$. Suppose not, $\exists T \ge t_1$ such that $D^{\alpha}u(t) < 0$. Since

$$D^{\alpha}\left[\left(\frac{1}{a(t)} D^{\alpha}(u(t))\right)\right] < 0, \quad t \ge t_1.$$

It is clear that

$$\frac{1}{a(t)} D^{\alpha}(u(t)) < \frac{1}{a(T)} D^{\alpha}(u(T)).$$

Therefore, by Lemma 1 we have

$$\frac{t^{1-\alpha}K'(t)}{d} \le D^{\alpha}u(t) < \frac{a(t)}{a(T)} D^{\alpha}u(T)$$

$$K'(t) < \frac{da(t)}{t^{1-\alpha}a(T)} D^{\alpha}u(T).$$

Integrate the above inequalities from t_0 to t, we have

$$K(t) < K(T) + \frac{dD^{\alpha}u(T)}{a(T)} \int_{t_0}^{t} \frac{a(s)}{s^{1-\alpha}} ds.$$

Letting $t \to \infty$, we get

$$\lim_{t\to\infty} K(t) \leq -\infty$$

which contradicts the fact that $D^{\alpha}(u(t)) \geq 0$ for $t \geq t_1$.

Define

$$\omega(t) = \frac{\delta(t)D^{\alpha}u(t)}{a(t)g(K(t))}. \tag{3}$$

Then, $\omega(t) > 0$ for $t \geq t_1$.

$$D^{\alpha}\omega(t) = \delta(t)\frac{f'(v(t))(-b(t)g(K) + r(t))}{g(K)}$$
$$- \frac{a(t)\delta(t)D^{\alpha}u(t)D^{\alpha}K(t)g'(K)}{(a(t)g(K))^2} + \frac{D^{\alpha}u(t)D^{\alpha}\delta(t)}{a(t)g(K)}.$$

Using assumptions I-III and (3) in the equation above, we have

$$D^{\alpha}\omega(t) \leq cmr(t)\delta(t) - m\delta(t)b(t) - da(t)M\frac{\omega^2 t}{\delta(t)} + t^{1-\alpha}\frac{\delta'(t)}{\delta(t)}\omega(t).$$

This implies that

$$\omega'(t) \leq \frac{m\delta(t)}{t^{1-\alpha}}(cr(t) - b(t)) - \frac{da(t)M}{t^{1-\alpha}\delta(t)}\omega^2(t) + t^{1-\alpha}\frac{\delta'(t)}{\delta(t)}\omega(t)$$
$$\leq \frac{m\delta(t)}{t^{1-\alpha}}(cr(t) - b(t)) + \left(\frac{t^{1-\alpha}\delta'^2(t)}{4da(t)M\delta(t)}\right). \tag{4}$$

Integrate the above inequality from t_0 to t, we have

$$\omega(t) \le \omega(t_0) - \int_{t_0}^{t} \left[\frac{m\delta(s)}{s^{1-\alpha}} (b(s) - cr(s)) - s^{1-\alpha} \left(\frac{\delta'}{2\sqrt{(dMa(s)\delta(s))}} \right)^2 \right] ds.$$

Taking the lim sup as $t \to \infty$, we have

$$\limsup_{t\to\infty} \omega(t) \le -\infty$$

which contradicts (2), the proof is complete. □

Corollary 1. *If assumptions I-III hold such that* (2) *is replaced with*

$$\limsup_{t\to\infty} \int_{t_0}^{t} m \frac{\delta(s)}{s^{1-\alpha}} (b(s) - cr(s)) ds = \infty \tag{5}$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{s^{1-\alpha}\delta'^2(t)}{4da(s)M\delta(s)} ds < \infty. \tag{6}$$

Then every solution of (1) *is oscillatory.*

In what follows, we introduce a class of functions P. Let

$$D_0 = \{(t, s) : t > s \ge t_0\}, \quad D = \{(t, s) : t \ge s \ge t_0\}.$$

The function $H \in C(D, \Re)$ is said to belong to the class of P if

(IV) $H(t, t) = 0$ for $t \ge t_0$, $H(t, s) > 0$ for $(t, s) \in D_0$

(V) $H(t, s)$ has a continuous and non-positive partial derivative $\frac{\partial H(t, s)}{\partial s}$ and

$$h(t, s) = \frac{\partial H(t, s)}{\partial s} + H(t, s) \frac{\delta'(s)}{\delta(s)}.$$

Theorem 2. *Suppose I-V hold and* $\exists$ *a positive function* $\delta \in C^{\alpha}([0, \infty), \Re_+)$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \delta(s)\left[m \frac{H(t, s)}{s^{1-\alpha}}(b(s) - cr(s)) - \frac{s^{1-\alpha} h^2(t, s)}{4dMa(s)H(t, s)}\right] ds = \infty. \quad (7)$$

Then every solution of (1) *is oscillatory.*

Proof. Let $(u(t), v(t))$ be a nonoscillatory solution of (1). Following the proof of Theorem 1, we obtain (4). Multiplying (4) by $H(t, s)$ and integrate from t_0 to t, we have

$$\int_{t_0}^{t} H(t, s)\omega'(s) ds \le \int_{t_0}^{t} H(t, s) \frac{m\delta(s)}{s^{1-\alpha}}(cr(s) - b(s)) ds$$

$$- \int_{t_0}^{t} H(t, s) \frac{da(s)M}{s^{1-\alpha}\delta(s)} \omega^2(s) ds$$

$$+ \int_{t_0}^{t} H(t, s) s^{1-\alpha} \frac{\delta'(s)}{\delta(s)} \omega(s) ds,$$

simplifying, we arrive at

$$\int_{t_0}^{t} \delta(s)\left[H(t, s) \frac{m}{s^{1-\alpha}}(b(s) - cr(s)) - \frac{s^{1-\alpha} h^2(t, s)}{4da(s)MH(t, s)}\right] ds \le H(t, t_0)\omega(t_0),$$

divide the above inequality by $H(t, t_0)$ and take the limit supremum as $t \to \infty$, we get

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)}$$

$$\times \int_{t_0}^{t} \delta(s)\left[H(t, s) \frac{m}{s^{1-\alpha}}(b(s) - cr(s)) - \frac{s^{1-\alpha} h^2(t, s)}{4da(s)MH(t, s)}\right] ds \le \omega(t_0) < \infty$$

which contradicts (7). The proof is complete.

Corollary 2. *If assumptions I-V hold such that* (7) *is replaced with*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \delta(s)\left[H(t, s) \frac{m}{s^{1-\alpha}}(b(s) - cr(s))\right] ds = \infty \quad (8)$$

and

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{s^{1-\alpha}h^2(t,\, s)}{4da(s)MH(t,\, s)}ds < \infty. \tag{9}$$

Then every solution of (1) *is oscillatory.*

Theorem 3. *Suppose I-V hold such that*

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^{2n-1}}$$

$$\times\int_{t_0}^{t}(t-s)^{2n-1}\left[m\frac{\delta(s)}{s^{1-\alpha}}(b(s)-cr(s))-\frac{s^{1-\alpha}}{4Ma(s)}\left(\frac{\delta'(s)}{\delta(s)}-\frac{(2n-1)}{t-s}\right)^2\right]ds = \infty. \tag{10}$$

Then every solution of (1) *is oscillatory.*

Proof. Let $(u(t),\, v(t))$ be a nonoscillatory solution of (1). Following the proof of Theorem 1, we obtain (4). Multiply (4) by $(t-s)^{2n-1}$ and integrate from t_0 to t, we have

$$\int_{t_0}^{t}(t-s)^{2n-1}\omega'(s)ds \le \int_{t_0}^{t}(t-s)^{2n-1}\frac{m\delta(s)}{s^{1-\alpha}}(cr(s)-b(s))ds$$
$$-\int_{t_0}^{t}(t-s)^{2n-1}\frac{da(s)M}{s^{1-\alpha}\delta(s)}\omega^2(s)ds$$
$$+\int_{t_0}^{t}(t-s)^{2n-1}s^{1-\alpha}\frac{\delta'(s)}{\delta(s)}\omega(s)ds,$$

simplifying the above inequality, we have

$$\int_{t_0}^{t}(t-s)^{2n-1}\left[\frac{m\delta(s)}{s^{1-\alpha}}(b(s)-cr(s))-\frac{s^{1-\alpha}}{4dMa(s)}\left(\frac{\delta'(s)}{\delta(s)}-\frac{(2n-1)}{t-s}\right)^2\right]ds$$
$$\le (t-t_0)^{2n-1}\omega(t_0),$$

divide through by $(t-t_0)^{2n-1}$ and take the lim sup as $t\to\infty$, we arrive at

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^{2n-1}}$$

$$\times \int_{t_0}^{t} (t-s)^{2n-1}\delta(s)\left[\frac{m}{s^{1-\alpha}}(b(s)-cr(s)) - \frac{s^{1-\alpha}}{4dMa(s)}\left(\frac{\delta'(s)}{\delta(s)} - \frac{(2n-1)}{t-s}\right)^2\right]ds$$

$\leq \omega(t_0) < \infty$

which contradicts (10). The proof is complete. □

4. Example

In this section, we give example to show the relevant of our results.

Consider the coupled system of fractional nonlinear differential equations

$$\left.\begin{aligned} &D^{2/3}(u(t)) = 2tv\exp(v(t)) \\ &D^{2/3}(v(t)) = -4t^3 g\left(\int_{t_0}^{t} u'(s)f'(u(s))\right)ds + \frac{6+t^2}{t} \end{aligned}\right\}. \quad (11)$$

By comparing (11) and (1), we deduce that

$$\left.\begin{aligned} &\alpha = 2/3,\ a(t) = 2t,\ b(t) = 4t^3,\ f(v(t)) = v\exp(v(t)); \\ &f'(v(t)) = v'\exp(v(t)) + v^2v'\exp(v(t)) = v'\exp(v(t))(1+v^2) \\ &r(t) = \frac{6+t^2}{t} \end{aligned}\right\}. \quad (12)$$

Let

$$\left.\begin{aligned} &\delta(t) = 1,\ \delta'(t) = 0,\ v(t) = t^2,\ \lambda = 2 \\ &f'(v(t)) = 2t(1+t^4)\exp(t^2) > 4 = m;\ H(t, s) = (t-s)^{\lambda} \\ &c = 1,\ g(u) = u(t) = t,\ g'(u) = 1 = M;\ h(t, s) = -2(t-s) \end{aligned}\right\} \quad (13)$$

substitute (12) and (13) into (2), we have

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{m\delta(s)}{s^{1-\alpha}}(b(s)-cr(s)) - s^{1-\alpha}\left(\frac{\delta'}{2\sqrt{(Ma(s)\delta(s))}}\right)^2\right]ds$$

$$= \limsup_{t\to\infty} \int_{2}^{t} \left[\frac{4}{s^{1/3}}4s^3 - \left(\frac{6+s^2}{s}\right) - 0\right]ds$$

$$= \limsup_{t\to\infty} \int_2^t (16s^{8/3} - 24s^{-4/3} - 4s^{2/3})ds = \infty.$$

This shows that Theorem 1 is satisfied. Hence, every solution of (11) is oscillatory. In the same way, we substitute (12) and (13) into (7), we have

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^t \delta(s)\left[m\frac{H(t, s)}{s^{1-\alpha}}(b(s) - cr(s)) - \frac{s^{1-\alpha}h^2(t, s)}{4Ma(s)H(t, s)}\right] ds$$

$$\limsup_{t\to\infty} \frac{1}{(t-2)^2} \int_2^t [4(t-s)^2(16s^{8/3} - 24s^{-4/3} - 4s^{2/3}) - 0.5s^{-2/3}]ds = \infty.$$

Suppose $n = 1$ and also substitute (12) and (13) into (10), we have

$$\limsup_{t\to\infty} \frac{1}{(t-2)} \int_2^t (t-s)\left[(16s^{8/3} - 24s^{-4/3} - 4s^{2/3}) - \frac{s^{1/3}}{8s(t-s)^2}\right] ds$$

$$\limsup_{t\to\infty} \frac{1}{(t-2)} \int_2^t (t-s)\left[(16s^{8/3} - 24s^{-4/3} - 4s^{2/3}) - 0.125\frac{s^{-2/3}}{(t-s)^{-2}}\right] ds.$$

These show that all the theorems (Theorems 1-3) are satisfied. Hence every solution of (11) is oscillatory.

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