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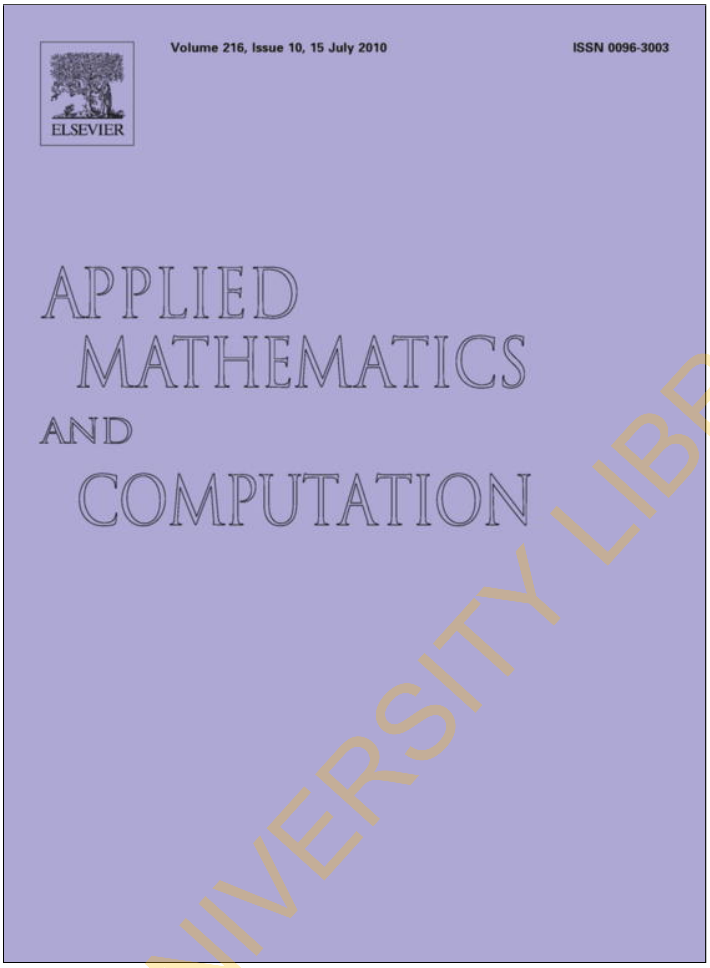

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Boundedness results for a certain third order nonlinear differential equation

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ABSTRACT

Sufficient conditions for the existence of solutions to boundedness and ultimate boundedness problems associated to a certain third order nonlinear differential equation are given by means of the Lyapunov's second method. The appropriate Lyapunov function is given explicitly. Our results complement some well known results on the third order differential equations in the literature.

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1. Introduction

The question addressed in this paper is related to the study of boundedness and ultimate boundedness of solutions which is very important in the theory and applications of nonlinear differential equations. In the actual literature, many works have been done on these properties of solutions; see for instance Reissig et al. [16], Rouche et al. [17] and Yoshizawa [24] which contain general theorems on the subject matter. Notable authors that have contributed to the qualitative properties of solutions of nonlinear third order differential equations include Ademola et al. [1] on uniform asymptotic stability of solutions; Afuwape [2,3] and Hara [14] on ultimate boundedness of solutions; Afuwape and Adesina [5], Andres [6], Bereketoğlu, and Györi [7], Ezeilo [8–12], Ezeilo and Tejumola [13], Swick [19], Tejumola [20] and Tunç [23] worked on boundedness of solutions. For the case when the considered third order equations are non-autonomous, we can mention the works of Qian [15], Swick [18] and Tunç [22] on asymptotic behaviour of solutions. Furthermore, Afuwape [4] and Tejumola [21] worked on periodic solutions.

Most of these works were done with the aid of Lyapunov functions. Unfortunately, with respect to our observation, these Lyapunov functions are either incomplete or contain signum functions. These we find too weak. Thus the purpose of this paper is to construct a complete Lyapunov function and use it to study boundedness (when $p = p(t,x,\dot{x},\ddot{x})$ in (1.1)) and ultimate boundedness of solutions of the third order nonlinear differential equation

$$\dddot{x} + f(\ddot{x}) + g(\dot{x}) + h(x) = p(t,x,\dot{x},\ddot{x}), \tag{1.1}$$

or its equivalent system of differential equations

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= z, \\ \dot{z} &= p(t,x,y,z) - f(z) - g(y) - h(x), \end{aligned} \tag{1.2}$$

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where $f, g, h \in C(\mathbb{R}, \mathbb{R})$, $p \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R} = (-\infty, \infty)$. It is assumed that the functions f, g, h and p depend only on the arguments displayed explicitly, and the dots, as usual, denote differentiation with respect to the independent variable t. We shall require that the derivative $h'(x) = \frac{dh(x)}{dx}$ exists and continuous, also the uniqueness of (1.1) or (1.2) will also be assumed. The results obtained in this work improve, generalize and complement existing results on third order nonlinear differential equations in the literature.

2. Preliminaries

Our notations shall follow those of Afuwape [3] and Hara [14]. Consider the system of the form

$$X' = F(t, X), \tag{2.1}$$

where $X \in \mathbb{R}^n, F : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function, $\mathbb{R}^+ = [0, \infty)$ and $\mathbb{R}^n$ is then Euclidean n-space.

Definition 2.1. The solutions of (2.1) are uniformly ultimately bounded for bound B, if there exists a $B > 0$ and if corresponding to any $\alpha_0 > 0$, there exists a $T(\alpha_0) > 0$ such that whenever $\|X_0\| = \|X(t, t_0, X_0)\| < \alpha_0$ then

$$\|X(t, t_0, X_0)\| < B \quad \text{for all } t_0 \geqslant 0 \quad \text{and} \quad t \geqslant t_0 + T(\alpha_0).$$

We now give a lemma which will play a major role in the proof of our results.

Lemma 2.2. *Suppose that there exists a Lyapunov function V(t, X(t)) defined on* $\mathbb{R}^+, \|X(t)\| \geqslant K$ *where K may be large, which satisfies the following conditions:*

(i) $a(\|X(t)\|) \leqslant V(t, X(t)) \leqslant b(\|X(t)\|)$, *where a(r), b(r) are continuous and increasing and* $a(r) \to \infty$ *as* $r \to \infty$;

(ii) $$V'_{(2.1)}(t, X(t)) \equiv \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, X(t) + F(t, X(t))) - V(t, X(t))] \leqslant -[c - \lambda_1(t)]V(t, X(t)) + \lambda_2(t)V^{\beta}(t, X(t))$$
$$(0 \leqslant \beta < 1), \tag{2.2}$$

where $c > 0$ *is a constant and* $\lambda_i \geqslant 0$ $(i = 1, 2)$ *are continuous functions satisfying*

$$\limsup_{(t,v) \to (\infty,\infty)} \frac{1}{v} \int_t^{t+v} \lambda_1(s) ds < c \tag{2.3}$$

and

$$\sup_{t \geqslant 0} \int_t^{t+1} \lambda_2(s) ds < \infty. \tag{2.4}$$

Then the solutions of (2.1) are uniformly ultimately bounded.

Proof. See Lemma 2.1 in [13] for $\beta = \frac{1}{2}$. □

3. Main results

Theorem 3.1. *Suppose that* a, b, b_1, c, δ_0 *are positive constants,* $p \equiv p(t)$ *and that*

(i) $h(0) = 0$, $\delta_0 \leqslant h(x)/x$, for all $x \neq 0$;
(ii) $h'(x) \leqslant c$ for all x;
(iii) $b \leqslant g(y)/y \leqslant b_1$, for all $y \neq 0$;
(iv) $a \leqslant f(z)/z$, for all $z \neq 0$;
(v) $\int_0^t |p(\mu)| d\mu \leqslant P_0 < \infty$ where P_0 is a positive constant.

Then for any given finite constants x_0, y_0, z_0 *there exists a constant* $D = D(x_0, y_0, z_0)$, *such that any solution* $(x(t), y(t), z(t))$ *of the system (1.2) determine by* $x(0) = x_0$, $y(0) = y_0$, $z(0) = z_0$ *for* $t = 0$, *satisfies*

$$|x(t)| \leqslant D, \quad |y(t)| \leqslant D, \quad |z(t)| \leqslant D, \tag{3.1}$$

for all $t \geqslant 0$.

Remark 3.2. When $f(\ddot{x}) = a\ddot{x}$, $g(\dot{x}) = b\dot{x}$, $h(x) = cx$ and $p(t, x, \dot{x}, \ddot{x}) = 0$, Eq. (1.1) reduces to a linear constant coefficient differential equation and conditions (i)–(v) of Theorem 3.1 reduce to the corresponding Routh–Hurwitz criterion $a > 0$, $ab > c$ and $c > 0$.

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The proofs of Theorem 3.1 and subsequent results depend on some certain fundamental properties of a continuously differentiable function $V(t) = V(x(t), y(t), z(t))$ defined by

$$2V(t) = 2a\int_0^x h(\xi)d\xi + 2\int_0^y g(\tau)d\tau + 2yh(x) + \alpha bx^2 + (\alpha + a^2)y^2 + z^2 + 2\alpha axy + 2\alpha xz + 2ayz, \quad (3.2)$$

where α is a positive fixed constant satisfying

$$0 < \alpha < b - \frac{c}{a}. \quad (3.3)$$

The Eq. (3.2) and its time derivatives satisfy some fundamental inequalities as will be seen later. In what follows, we shall state and prove some results that would be useful in the proof of the main result.

Lemma 3.3. *Under the hypotheses of Theorem 3.1 , there exist positive constants $D_i (i = 0, 1)$ such that for all $(x, y, z) \in \mathbb{R}^3$*

$$D_0(x^2(t) + y^2(t) + z^2(t)) \leqslant V(t) \leqslant D_1(x^2(t) + y^2(t) + z^2(t)). \quad (3.4)$$

Proof. We observe that the function in Eq. (3.2) can be rewritten as

$$2V(t) = V_1 + V_2,$$

where

$$V_1 = 2a\int_0^x h(\xi)d\xi + 2\int_0^y g(\tau)d\tau + 2yh(x)$$

and

$$V_2 = \alpha bx^2 + (\alpha + a^2)y^2 + z^2 + 2\alpha axy + 2\alpha xz + 2ayz.$$

In view of hypothesis (iii) in Theorem 3.1, $g(y) \geqslant by$ for all $y \neq 0$, thus

$$2\int_0^y g(\tau)d\tau + 2yh(x) \geqslant (by + h(x))^2 b^{-1} - b^{-1}h^2(x) \geqslant -b^{-1}h^2(x). \quad (3.5)$$

This is true since $(by + h(x))^2 \geqslant 0$ for all x, y. Moreover, hypotheses (i) and (ii) of Theorem 3.1 imply that

$$2a\int_0^x h(\xi)d\xi = 2b^{-1}\int_0^x (ab - h'(\xi))h(\xi)d\xi + b^{-1}h^2(x) \geqslant (ab - c)b^{-1}\delta_0 x^2 + b^{-1}h^2(x). \quad (3.6)$$

On combining the inequalities (3.5) and (3.6), we obtain

$$V_1 \geqslant (ab - c)b^{-1}\delta_0 x^2 \quad (3.7)$$

for all x. Furthermore, V_2 can be rewritten as

$$V_2 = XQ_0X^T,$$

where $X = (x \quad y \quad z), Q_0 = \begin{pmatrix} \alpha b & \alpha a & \alpha \\ \alpha b & \alpha + a^2 & a \\ \alpha & a & 1 \end{pmatrix}$ and $\det Q_0 = \alpha^2(b - \alpha) > 0$, since $b - \alpha > 0$ (which follows from (3.3)). Thus

$$V_2 \geqslant \alpha^2(x^2 + y^2 + z^2) \quad (3.8)$$

for all $(x, y, z) \in \mathbb{R}^3$ with $\alpha > 0$. On gathering the inequalities (3.7) and (3.8), the lower inequality in (3.4) is obtained. Now to obtain the upper inequality in (3.4), we proceed as follows. Since $h(0) = 0$, hypothesis (ii) of the Theorem 3.1 implies that $h(x) \leqslant cx$ for all $x \neq 0$. It follows from hypotheses (ii) and (iii) of the theorem that

$$V_1 \leqslant c(a + 1)x^2 + (b_1 + c)y^2, \quad (3.9)$$

$$V_2 \leqslant \alpha(a + b + 1)x^2 + (\alpha + a)(a + 1)y^2 + (\alpha + a + 1)z^2. \quad (3.10)$$

On gathering the estimates (3.9) and (3.10), the upper inequality in (3.4) follows immediately. □

From (3.2) it is clear that $V(0,0,0) = 0$, the lower inequality in the inequalities (3.4) implies, $V(x,y,z) > 0$ as $x^2 + y^2 + z^2 \neq 0$, hence it follows that

$$V(x, y, z) \to \infty \quad \text{as } x^2 + y^2 + z^2 \to \infty. \quad (3.11)$$

Inequality (3.4) together with (3.11) established condition (i) of the Lemma 2.2.

Lemma 3.4. *Under the hypotheses of the theorem, there are positive constants D_i, $(i = 2, 3, 4, 5)$ such that if $(x(t), y(t), z(t))$ is any solution of the system (1.2), then*

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$$\dot{V}_{(1.2)} = \frac{d}{dt}V(x(t),y(t),z(t)) \leqslant -(D_2x^2 + D_3y^2 + D_4z^2) + D_5(|x| + |y| + |z|)|p(t)|. \tag{3.12}$$

Proof. Along any solution $(x(t),y(t),z(t))$ of the system (1.2), it follows from the Eq. (3.2) that

$$\dot{V}_{(1.2)}(t) = -\alpha xh(x) - (ayg(y) - y^2h'(x)) + \alpha(g(y) - by) - (\alpha x + ay + z)(f(z) - az) + (\alpha x + ay + z)p(t) + \alpha YQ_1Y^T, \tag{3.13}$$

where $Y = (y \quad z)$, $Q_1 = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$, and $\det Q_1 = -1$. In view of hypotheses (i)–(iv), we have that

$$\dot{V}_{(1.2)}(t) \leqslant -\frac{1}{2}\alpha\delta_0x^2 - \frac{7}{8}(\alpha + ab - c)y^2 - \frac{1}{2}\alpha z^2 - W_j + (\alpha x + ay + z)p(t) \quad (j = 1,2,3), \tag{3.14}$$

where

$$W_1 = \alpha\left(\frac{1}{4}\delta_0x^2 + (g(y) - by)x + \frac{1}{16\alpha}(\alpha + ab - c)y^2\right); \tag{3.15}$$

$$W_2 = \alpha\left(\frac{1}{4}\delta_0x^2 + (f(z) - az)x + \frac{1}{4}z^2\right); \tag{3.16}$$

$$W_3 = a\left(\frac{1}{16a}(\alpha + ab - c)y^2 + (f(z) - az)y + \frac{\alpha}{4a}z^2\right). \tag{3.17}$$

Using the Eqs. (3.15)–(3.17), and taking into consideration the following inequalities

$$(g(y) - by)^2 < \frac{\delta_0(\alpha + ab - c)}{16\alpha}y^2; \tag{3.18}$$

$$(f(z) - az)^2 < \frac{\delta_0}{4}z^2; \tag{3.19}$$

$$(f(z) - az)^2 < \frac{\alpha(\alpha + ab - c)}{16a^2}z^2; \tag{3.20}$$

we have that

$$W_1 \geqslant \frac{\alpha}{16}\left(2\sqrt{\delta_0}|x| - \sqrt{\frac{\alpha + ab - c}{\alpha}}|y|\right)^2 \geqslant 0 \quad \text{for all } x,y; \tag{3.21}$$

$$W_2 \geqslant \frac{\alpha}{4}\left(\sqrt{\delta_0}|x| - \sqrt{\frac{\alpha}{a}}|z|\right)^2 \geqslant 0 \quad \text{for all } x,z; \tag{3.22}$$

$$W_3 \geqslant \frac{a}{16}\left(\sqrt{\frac{\alpha + ab - c}{a}}|y| - 2\sqrt{\frac{\alpha}{a}}|z|\right)^2 \geqslant 0 \quad \text{for all } y,z. \tag{3.23}$$

On making use of the estimates (3.21)–(3.23) in (3.14), we obtain

$$\dot{V}_{(1.2)}(t) \leqslant -\frac{1}{2}\alpha\delta_0x^2 - \frac{7}{8}(\alpha + ab - c)y^2 - \frac{1}{2}\alpha z^2 + \max(\alpha, a, 1)(|x| + |y| + |z|)|p(t)|, \tag{3.24}$$

and this completes the proof of the Lemma 3.4. □

At last we shall now give the proof of the Theorem 3.1.

Proof of Theorem 3.1. Let $(x(t),y(t),z(t))$ be any solution of (1.2), then from (3.24), it follows that

$$\dot{V}_{(1.2)}(t) \leqslant \delta_1(3 + x^2 + y^2 + z^2)|p(t)|,$$

where $\delta_1 \equiv \max(\alpha, a, 1)$. Now, from the inequalities (3.4), we obtain

$$\dot{V}_{(1.2)}(t) - \delta_2V(t)|p(t)| \leqslant \delta_2|p(t)|$$

where $\delta_2 = \max(3\delta_1,\ \delta_1D_0^{-1})$. Multiplying each side by the integrating factor $\exp\left(-\delta_2\int_0^t|p(\mu)d\mu|\right)$, and integrate from 0 to t to obtain

$$V(t) \leqslant V(0)e^{\delta_2P_0} + e^{\delta_2P_0} - 1 \equiv \delta_3(x_0, y_0, z_0),$$

since $V(0) = V(x_0,y_0,z_0)$. In view of the inequalities (3.4) we have

$$x^2 + y^2 + z^2 \leqslant \delta_4,$$

where $\delta_4 = \delta_3D_0^{-1}$, this verifies the inequalities (3.1) with $D \equiv \delta_4^{1/2}$. This completes the proof of the Theorem 3.1.

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Our next result is on the ultimate boundedness of solutions to the Eq. (1.2). □

Theorem 3.5. *Suppose that a, b, b_1, c, δ_0 are positive constants and that*

(i) Conditions (i)–(iv) of the Theorem 3.1 hold;
(ii) for all $(x,y,z)\in\mathbb{R}^3$ and $0\leqslant t\in\mathbb{R}^+$ there are nonnegative continuous functions $p_1(t)$ and $p_2(t)$ such that

$$|p(t,x,y,z)|\leqslant p_1(t)+p_2(t)(|x|+|y|+|z|)\quad and\quad |x|+|y|+|z|\geqslant\rho\quad(\rho>0,)\tag{3.25}$$

where $\sup\int_t^{t+1}p_1(\mu)d\mu<\infty$ *and there is* $\epsilon>0$ *such that* $0\leqslant p_2(t)<\epsilon$.

Then the solution (x(t), y(t), z(t)) of (1.2) is uniformly ultimately bounded.

Proof of Theorem 3.5. Consider the equivalent system (1.2) and the Lyapunov function $V(t)$ as defined in (3.2). If the inequalities (3.4) hold for $V(x,y,z)$, it follows that

$$V(x,y,z)\to\infty\quad\text{as } x^2+y^2+z^2\to\infty.\tag{3.26}$$

From the inequalities (3.4) and relation (3.26), condition (i) of Lemma 2.2 is established.

Next, we shall show that condition (ii) of Lemma 2.2 holds for the system (1.2). To see this, conclusion of Lemma 3.4 can be revised as follows

$$\begin{aligned}\dot V_{(1.2)}(t)&\leqslant-\min(D_1,D_2,D_3)(x^2+y^2+z^2)+D_5(|x|+|y|+|z|)|p(t,x,y,z)|\\&\leqslant-\delta_5(x^2+y^2+z^2)+D_5(|x|+|y|+|z|)^2p_2(t)+D_5(|x|+|y|+|z|)p_1(t)\\&\leqslant-\delta_5(x^2+y^2+z^2)+3D_5(x^2+y^2+z^2)p_2(t)+\sqrt{3}D_5(x^2+y^2+z^2)^{1/2}p_1(t),\end{aligned}$$

provided $|x|+|y|+|z|\geqslant\rho$. Using the inequalities (3.4), for all $(x,y,z)\in\mathbb{R}^3$ and $0\leqslant t\in\mathbb{R}^+$, we have that

$$\dot V_{(1.2)}(t)\leqslant-\left[\delta_5D_1^{-1}-3D_2^{-1}D_5p_2(t)\right]V(x,y,z)+D_5\sqrt{3D_0^{-1}V(x,y,z)}p_1(t).$$

Let

$$p_2(t)=\limsup_{(t,v)\to(\infty,\infty)}\frac{1}{v}\int_t^{t+v}p_1(\mu)d\mu<3^{-1}\delta_5D_1^{-1}D_2D_5^{-1}.$$

Thus, choose $c=\delta_5D_1^{-1}$, $\lambda_1(t)=3D_2^{-1}D_5p_2(t)$, $\lambda_2(t)=D_5\sqrt{3D_0^{-1}}p_1(t)$ and $\beta=1/2$, condition (ii) of Lemma 2.2 is established. This completes the proof of the Theorem 3.5. □

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