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Investigating Duffing Oscillator using Bifurcation Diagrams

T.A.O. Salau and O.O. Ajide

Department of Mechanical Engineering, University of Ibadan, Nigeria
E-mail: tao.salau@mail.ui.edu.ng, getjidefem2@yahoo.co.uk

Abstract

This paper investigates the dynamical behaviour of a duffing oscillator using bifurcation diagrams. There has been growing interest and challenges in engineering dynamics to characterize dynamical systems that are chaotic using bifurcation diagrams. The relevant second order differential equations using Runge-Kutta method were solved for ranges of appropriate parameters. The solutions obtained were used to produce the bifurcation diagrams using Microsoft excel 2007. Since an average estimate of $\delta = 4.668$ from the bifurcation diagrams produced is an approximate value of the Feigenbaum constant as widely reported in the literatures, it can be deduced that the bifurcation diagrams conforms to the expected results. While the bifurcation diagrams revealed the dynamics of the duffing oscillator, it also shows that the dynamics depend strongly on the initial conditions.

Keywords: Bifurcation, Duffing oscillator, chaotic dynamical behaviour, *Feigenbaum constant.*

Introduction

Nonlinear science and chaos is a field of growing interest to scientists due to its usefulness in such diverse fields as Physics, Biology, Engineering, Medicine and Chemistry among others. There has been growing interest and challenges in engineering dynamics to characterize dynamic systems that are chaotic. Bifurcation diagram has been of great help in diagnosing response of complex dynamics systems to a tunable parameter and as an aid in computation of a Universal constant called Feigenbaum constant.

In dynamics, a change in the number of solutions to a differential equation as a parameter is varied is called a bifurcation (*Thompson* and *Stewart*, 1986). Bifurcation is also described as a record of change in behaviour of a dynamical system as parameter changes. *Julyan* and *Oreste* (1992) emphasized that one of the major ways of investigating the dynamics of a continuous time system by differential equation is the use of Runge-Kutta methods in developing bifurcation diagram. Many researchers have contributed to the use of bifurcation in the study of chaotic dynamic systems. *Han et al* (1995) developed a model and used bifurcation diagram as a tool for investigating chaotic phenomena in vibratory ball milling system. *McDonough* (2004) applied bifurcation in the analysis of low-dimensional models of turbulent combustion. The recent developments in understanding the nature of chaos is making it possible to tackle it in real-world systems. A framework to model real-world chaotic systems from their short, noisy, observed data, to understand their behavioural changes with respect to time and parametric space has been developed by *Farugi* and *Kumar* (2005). The study has been performed in qualitative analysis by constructing the bifurcation diagrams. *Joseph* (2008) developed a model and bifurcation diagrams of chaotic frequency scaling in a coupled oscillator model for free rhythmic actions.

Duffing Oscillator is one of the most intensively studied systems in dynamics, and it is employed as models of various physical and engineering situations such as Josephon junctions, optical bistability, plasma oscillators, buckled beam, ship dynamics, vibration isolators and electrical circuits (*Jun Yu* and *et al). Wagg* and *Adhikari* (2006) studied the dynamics of Duffing oscillator with an exponential non-viscous model.

Extensive work has not been done using bifurcation diagrams in investigating the dynamics of this Duffing oscillator.

Model Descriptions

For $\beta > 0$, the Duffing oscillator can be interpreted as a forced oscillator with a spring whose restoring force is written as

$$F = -\beta x - \alpha^3 \tag{1}$$

where $\alpha > 0$, this spring is called a hardening spring and when $\alpha > 0$, it is called a softening spring.

For $\beta < 0$, the Duffing oscillator describes the dynamics of a point mass in a double wall potential, and it can be regarded as a model of a periodically forced steam beam which is deflected towards the two magnets as shown in figure 1.

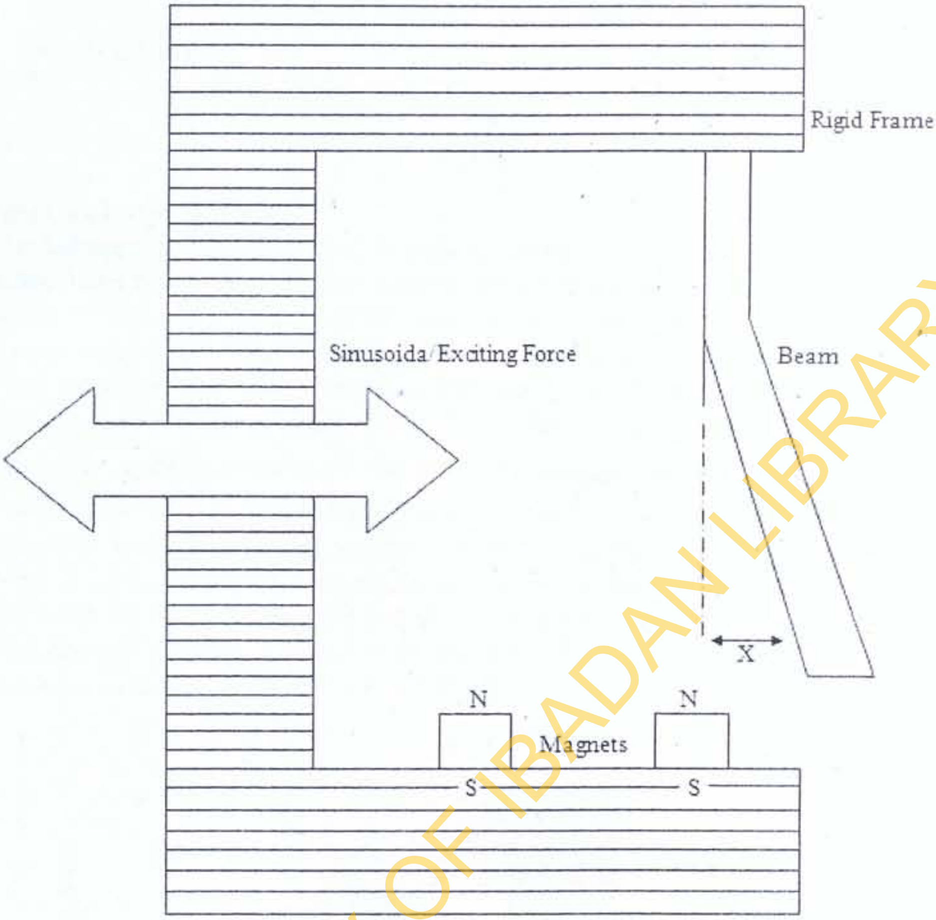

Figure 1: Duffing Oscillator.

According to Takashi (2008), Duffing oscillator can be described as a periodically forced oscillator with a non-linear elasticity governed by an equation.

$$\ddot{x} + k\dot{x} + \beta x + \alpha x^3 = P\cos(wt) \qquad (2)$$

According to *Salau* (2007) as coined from chaotic dynamics by *Gregory, L.B* and *Jerry, P.G* (1990), the general equation of a damped and forced Duffing dynamical system are given as

$$\ddot{x} + k\dot{x} \pm \alpha x\left(1 - x^2\right) = F(t) \qquad (3)$$

$$\ddot{x} + k\dot{x} \pm \alpha x\left(1 - x^2/2\right) = F(t) \qquad (4)$$

Putting $\alpha = 1$ and $F(t) = P_o \cos(wt)$, the governing equation employed for this case study is given as

$$\ddot{x} + k\dot{x} - x + 0.5x^3 = P_o \cos(wt) \quad (5)$$

Numerical Simulations

Numerical approach is employed in solving equation 5. Numerical solutions were obtained using fourth-order Runge-Kutta and FORTRAN 90 Source codes. According to *Salau* (2007), chaotic behaviour may be observed using initial conditions of parameter values $K = 0.4$ and $\omega = 0.5$. Figure 2 describes the bifurcation diagram of a Duffing oscillator when the damping coefficient $k = 0.35$, angular frequency ω=0.5 and the number of slices, Nslice = 200. The initial conditions used is set at $x_0 = 0.2$ and $y_0 = 0.2$. A stable solution of five thousand cycles at constant time step of 0.001 was achieved after 30 complete transition cycles $(Ns = 15,\ Ne = 15)$.

The first periodic doubling is observed at forcing strength 1.690N and angular velocity of 2.05291 rad/s. An infinitesimal periodic window is observed in the range $1.659N < P < 1.736N$. A pair of period-doubling route to chaos occurs at $1.690N$ forcing strength P with 2.07452 rad/s angular velocity. The first region of chaotic behaviour is in the range $1.831N < P < 2.111N$.

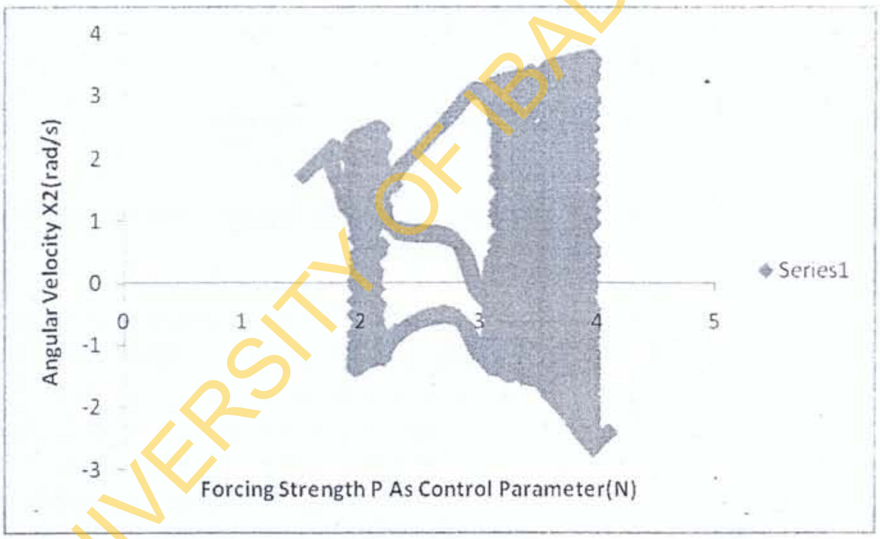

Figure 2: Bifurcation Diagram of Duffing Oscillator Model (K=0.35, ω=0.5, Nslice=200).

A wide periodic window is thereafter observed in the range $2.111N < P < 3.135N$. The second but a very wide chaotic region is observed in the range $3.135N < P < 3.913N$ as a result of three pairs of period-doubling route to chaos which occurs at forcing strength P of $2.154N$ with angular velocity of 1.76177 rad/s, 2.073N with angular velocity of 0.93961 rad/s and $2.073N$ with angular velocity 0.70286 rad/s.

The interpretations of this dynamic behaviour are that the ranges $1.831N < P < 2.111N$ and $3.135N < P < 3.913N$ are chaotic regions. Forcing strengths in these ranges should be avoided or ignored if chaos is not desired in the Duffing oscillator system at the set initial conditions. However, if chaotic phenomenon will be of great merit to the system, such ranges of forcing strength should be considered.

Figure 3: Bifurcation Diagram of Duffing Oscillator Model (K=0.4, ω=0.5,N_{slice}=50).

Figure 3 is obtained when the damping coefficient is increased from $k = 0.35$ to $k = 0.4$. The initial conditions slightly changed to $x_0 = 0.1, y_0 = 0.1$, N_{slice} = 50 $w = 0.5$. A different dynamical behaviour of Duffing oscillator is experienced. The bifurcation diagram in figure 3 is produced when the number of transition cycles $Ns = 16$ and number of cycles examined, $Ne = 16$. The first pair of bifurcation occurs at forcing strength $P = 1.735N$ with the angular velocity of 1.9827 rad/s. A small periodic window is experienced in the range $1.745N < P < 1.952N$. Two pairs of period-doubling route to chaos occurs at $2.125N$ and $2.120N$ forcing strengths $P's$. The first period-doubling leads to chaotic behaviour in the range $1.952N < P < 2.149N$. The widest periodic window is observed in the range $2.149N < P < 2.961N$.

The three pairs of period-doubling occurs at $2.763N$, $2.751N$ and $2.738N$ forcing strengths $P's$ leading to a chaotic region in the range $2.951N < P < 3.121N$. Another periodic behaviour is observed in the range $3.114N < P < 3.387N$ due to four pairs of period-doubling. The last chaotic region is observed in the range 3.366N<P<3.452N.The inference that can be drawn from this analysis is that the ranges $1.745N < P < 1.952N$, $2.149N < P < 2.961N$ and $3.114N < P < 3.387N$ should be adopted when there is need to ignore the phenomenon of chaos at this set initial conditions. When chaotic behaviour is desired in the system, the forcing strengths in the ranges $1.952N < P < 2.149N$, 2.951N<3.121N and $3.366N < P < 3.452N$ should be employed.

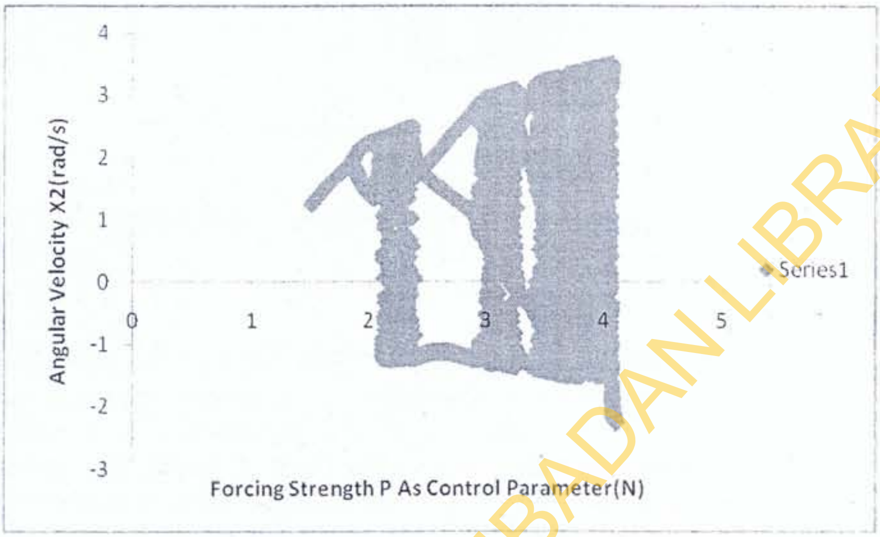

Figure 4: Bifurcation Diagram of Duffing Oscillator Model (K=0.45, ω=0.5, N_{slice}=100)

Using $x_0 = 0.1$ and $y_0 = 0.1$ as in figure 3 by increasing the damping coefficient to 0.45, putting N_{slice} = 100, $Ns = 20$ and $Ne = 20$, a unique dynamical behaviour is experienced between $1.501N$ and $4.069N$ forcing strengths as shown in the bifurcation diagram of figure 4. A pair of period-doubling occurs at forcing strength $P = 1.789N$ with angular velocity $X_2 = 1.80794$ rad/s.The first chaotic region occurs in $2.071N < P < 2.334N$. Immediately after this is a wide period-doubling route to chaos occurs at $P = 2.289N$ with 0.79896 rad/s angular velocity. This leads to a chaotic region in the range $2.907N < P < 3.195N$. Thereafter is a periodic window in the range $3.195N < P < 3.444N$. The largest region of chaotic behaviour is observed as a result of four pairs of period-doubling route to chaos which occurs at forcing strengths of $P = 3.193N, 3.179N$, 3.204N and 3.163N . The range of this largest chaotic phenomenon is $3.444N < P < 4.069.N$ .The deductions here is that when a stable, predictable and non-chaotic behaviour is desired, the ranges $2.301N < P < 2.921N$ and $3.195N < P < 3.444N$ should be employed.

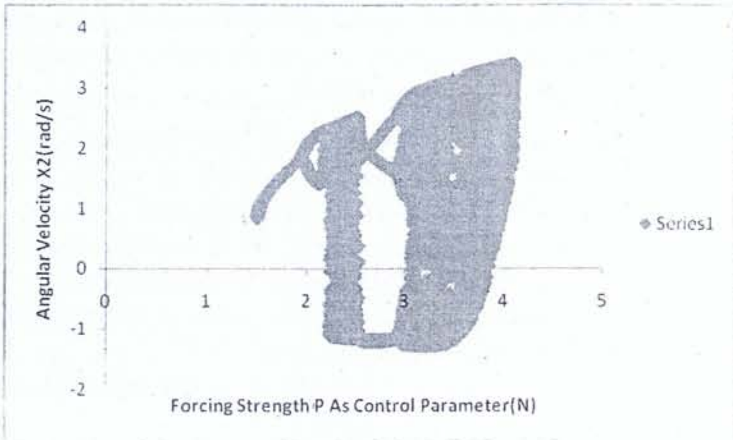

Figure 5: Bifurcation Diagram of Duffing Oscillator Model (K=0.5, ω=0.5,N_{slice}=200)

With $x_0 = 0.1$, $y_0 = 0.1$ and $w = 0.5$ as used in figure 4, an interesting and distinct dynamical behaviour is experienced as shown in figure 5 when damping coefficient K is increased to 0.5, N_{slice} = 200, $Ns = 15$ and $Ne = 15$. The first bifurcation pair occurs at $P = 1.889N$ and 1.7727 rad/s angular velocity, the first region of chaos is in the range $1.966N < P < 2.431N$ This is followed by the major periodic window in the range $2.431N < P < 2.962N$ while three pairs of period-doubling leads to the major chaotic behaviour in the range $2.962N < P < 4.051N$ The interpretation of this analysis is that the ranges $1.966N < P < 2.431N$ and $2.962N < P < 4.051N$ should be given a high consideration when chaotic phenomenon is highly advantageous.

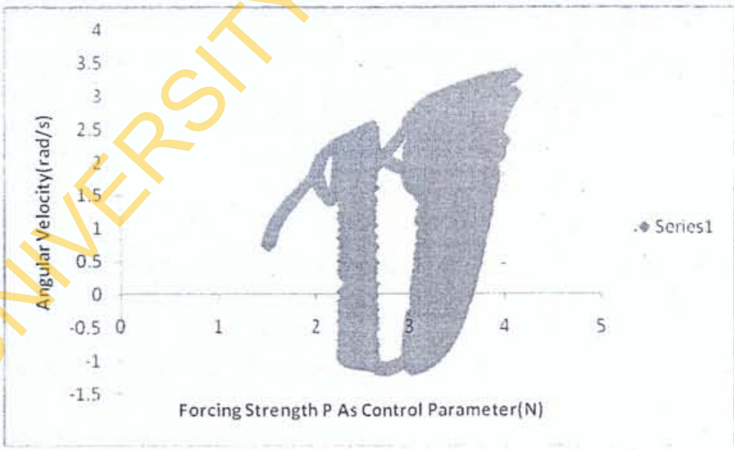

Figure 6: Bifurcation Diagram of Duffing Oscillator Model (K=0.53, ω=0.5,N_{slice}=100)

The sensitivity to initial conditions which is a core property of chaotic behaviour is revealed when the initial conditions changes to $x_0 = 0.1$ and $y_0 = 0.2$. other parameters are put as $K = 0.53$, $w = 0.5$ and N_{slice} = 100. The number of cycles sacrificed (Ns) is 30 and also 30 complete cycles are examined (Ne). A pair of period-doubling begins at $P = 1.964N$ and angular velocity of 1.77388 rad/s. The first chaotic region is experienced in the range $2.285N < P < 2.524N$ and followed by a wide periodic window in the range $2.516N < P < 3.086N$. The major and well pronounced chaotic behaviour is experienced in the range $3.106N < P < 4.043N$ due to three pairs of period-doubling which occurs at forcing strengths of $P = 2.876N$, $P = 2.755N$ and $P = 2.771N$. From this analysis, it can be deduced that the only reliable region for a non-chaotic phenomenon is $2.516N < P < 3.086N$.

Results Validation

The bifurcation diagrams obtained for Duffing oscillator model are confirmed using Feigenbaum constant (σ) as stated in the equation 6.

$$\delta = \lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} \qquad (6)$$

It is estimated using figure generated 6 when $x_0 = 0.1$, $x_0 = 0.2$, $K = 0.53$, $w = 0.5$ and N_{slice} = 100 as a representative illustration for all other Duffing bifurcation diagrams produced in this paper. The estimation is done as follows:

δ =(2.9720 – 2.0160)/(3.1768-2.9720)

δ = (0.9560)/(0.2048)

δ = 4.668

Since δ = 4.668 calculated from the bifurcation diagrams produced is an approximate value of the Feigenbaum constant ($\delta = 4.669201609102990 9....$) as widely reported in the literatures, it can be inferred that the bifurcation diagrams validates or conforms to the expected results.

Conclusions

The results of this study have shown that bifurcation diagram is a resourceful instrument for global view of the dynamics of Duffing oscillator system over a range of control parameter. It gives an advantage of simultaneous comparison of both periodic and chaotic behaviour of dynamical systems.

The results also revealed and confirmed that sensitivity to initial conditions is a principal property of all chaotic dynamical systems.

All bifurcation diagrams produced in this work reveals that even a very tiny alteration in the initial conditions gives a unique and interesting bifurcation diagrams with distinct dynamical behaviours.Findings also reveal that a slight change in any of the adjustable parameters in chaotic governing models generated distinct dynamical behaviours.

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Notations

β : Displacement Damping Coefficient

α : Spring Softening or Hardening Constant

P_o : Forcing Strength

w : Angular Frequency

t : Period in Seconds

K : Damping Coefficient

x_o : Initial Condition of X

y_o : Initial Condition of Y

N_{slice}: Numbers of Slice

Ns : Number of Cycles Sacrificed

Ne : Number of Cycles Examined

δ : Feigenbaum Constant

X_2 : Angular Velocity

F (t): Damped Force

N: Unit of force in Newton

Appendix

Bifurcation Diagram of a Duffing Oscillator

```
Implicit real *8(a-h, o-z)
Dimension x1(2),x2(2),Rt(4),Rk(4),Rg(4),Rf(4)
Open (unit=1, file='jideE.out')
Open (unit=2, file='jideENslice.out')
Pi2=6.0*acos (0.5)
Write (*,*) Pi2
Write (*,*)'Enter Damping Coefficient'
Read (*,*) Dampf
Write (*,*)'Enter Initial conditions and wf, Nslice'
Read (*,*) Xo, Yo, wf, Nslice
Fw=wf/pi2
TP=1.0/fw
Deltat=Tp/float (Nslice-1)
```

```
Deltat6=Deltat/6.0
Tole=0.000001
Write (*,*)'Enter No of Run away cycles and to be examine'
Read (*,*) Ns, Ne
Pp=1.5
Fp=4.1
Step=0.001
Ncut=int ((fp-pp)/step)
Do 30 kk=1, Ncut
Tt=0
X1(1) =xo
X2(1) =yo
Pp=pp+step
```

The Real Experiments

```
Do 20 I1=1, NE+Ns
Do 20 I2=1, Nslice
Rt (1)=tt
Rt (2) =TT+deltat*0.5
Rt (3) =Rt (2)
Rt (4) =Rt (1) +deltat
Do 15 I=1, 4
If (i.eq.1) then
Rk (i) =x1(1)
Rg (i) =x2(1)
Else
If (i.eq.4) then
Rk (i) =x1(1) +deltat*Rg (i-1)
Rg (i) =x2(1) +deltat*Rf (i-1)
Else
Rk (i) =x1(1) +Rg (i-1)*deltat*0.5
Rg (i) =x2(1) +Rf (i-1)*deltat*0.5
Titaf=wf*Rt (i)
```

```
    Titak=rk (i)
    Rf (i) =Pp*cos (titaf) +Rk (i)-0.5*(Rk (i) **3)-Dampf*Rg (i)
    Endif
    Endif
     15 Continue
    X1(2) =x1(1) +deltat6*(Rg (1) +2.0*(Rg (2) +Rg (3)) +Rg (4))
    X2(2)=x2(1)+deltat6*(Rf(1)+2.0*(Rf(2)+Rf(3))+Rf(4))
    TT=Rt (4)
```

The next if statements ensure stable results are reported!

```
    Titaf=mod (wf*tt, pi2)
    If (I1.gt.Ns.and.kk.eq.1.and.titaf.le.tole) write (*,*) titaf, tole
    If (I1.gt.Ns.and.titaf.le.tole) Write (1, 25) pp, x1(2), pp, x2(2)
    If (I1.gt.Ns.and.I2.eq.Nslice) Write (2, 25) pp, x1(2), pp, x2(2)
    X1(1) =x1(2)
    X2(1) =x2(2)
20  Continue
30  Continue
25  Format (4(f10.5, 2x))
    Stop
    End
```